Well-posedness of the frequency permutation problem in sparse filter estimation with lp minimization

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Well-posedness of the frequency permutation problem in sparse filter estimation with ℓ_p minimization

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***Abstract*—A well-known issue in blind convolutive source separation is that the sources and filters are at best identifiable up to an arbitrary scaling and permutation at each frequency bin. We propose to exploit the sparsity of the filters as a consistency measure for correcting such permutations. We show that the permutation is well-posed, up to a global permutation, under appropriate sparsity hypotheses on the filters. A greedy combinatorial algorithm is proposed for permutation recovery. Its empirical performance shows that the time-domain sparsity of the filters allows to recover permutations much beyond theoretical predictions.**

I. CONTEXT

Let $x_i[t]$ be M mixtures of N source signals $s_j[t]$, resulting from the convolution with a filter $a_{ij}[t]$ of length L such that:

$$x_i[t] = \sum_{j=1}^{N} (a_{ij} \star s_j)[t], \quad 1 \leq i \leq M. \tag{1}$$

We consider the problem of estimating the matrix of filters $\mathbf{A} = (a_{i,j})$ from the mixtures, without knowledge about the sources. A standard approach is to formulate the problem in the Fourier domain: one needs to estimate $a_{ij}[\omega]$. This suffers from a well known ambiguity : without further assumption on either $a_{ij}[t]$ or $s_j[t]$, one can at best hope to find an estimation $\tilde{\mathbf{A}} = (\tilde{a}_{i,j})$ where for every frequency $\omega \leq L$ we have

$$\tilde{a}_{i,j}[\omega] = \lambda_j[\omega] a_{i\sigma_\omega(j)}[\omega], \tag{2}$$

with λ_j a scaling ambiguity and σ_ω a permutation ambiguity. Several methods [1] to exploit properties of either $\mathbf{S}$ or $\mathbf{A}$ solve these. Our focus here is on the use of the sparsity of $\mathbf{A}$ in the time domain to find $\sigma_1 \ldots \sigma_L \in \mathfrak{S}_N$, assuming the scaling $\lambda \in \mathbb{C}^L$ is solved. Of course we can at best hope to obtain uniqueness up to a global permutation of the columns of $\mathbf{A}$. We exploit [2, Th.6.2a] the ℓ_p quasi-norm $\|\mathbf{A}\|_p^p := \sum_{ijt} |a_{ij}[t]|^p$, $0 \leq p \leq 1$, as a consistency measure to solve the permutations.

II. THEORETICAL GUARANTEES

If the filters a_{ij} have disjoint supports, without further sparsity hypothesis, we show that permutations can only increase the l_p norm.

Theorem 1 ([3]): Let $\Gamma_{ij} \subset \{1, \ldots, L\}$ be the time domain support of a_{ij}. Suppose that for all i and $j_1 \neq j_2$ we have $\Gamma_{i,j_1} \cap \Gamma_{i,j_2} = \emptyset$. Then for $0 \leq p \leq 1$ we have $\|\mathbf{A}\|_p^p \leq \|\tilde{\mathbf{A}}\|_p^p$.

To obtain uniqueness guarantees, we now introduce assumptions on the sparsity $k := \max_{i,j} \|a_{i,j}\|_0$. We measure the permutation error for $0 \leq p \leq 1$ with

$$\Delta_p := \min_{\pi \in \mathfrak{S}_N} \max_{i,j} \| \left\{ a_{i\pi(j)}[\omega] - \tilde{a}_{ij}[\omega] \right\}_{1 \leq \omega \leq L} \|_p. \tag{3}$$

For sparse filters, the true filters are the sparsest among all filters incurring sufficiently few permutations. The skilled reader will rightly sense the role of the ℓ_0 Fourier-Dirac uncertainty principle [4] in the following result.

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Theorem 2 ([3]): (i) If $1 \leq \Delta_0 \leq L/2k$, then $\|\mathbf{A}\|_0 \leq \|\tilde{\mathbf{A}}\|_0$.
(ii) If $\|\mathbf{A}\|_0 \geq \|\tilde{\mathbf{A}}\|_0$, then $\Delta_0 \geq \frac{L}{2k}$.

For prime L, the results hold with $L + 1 - 2k$ instead of $\frac{L}{2k}$. The equality case implies that the filters are pathologically related to Dirac combs of step $\frac{L}{2k}$.

III. A COMBINATORIAL ALGORITHM

We perform minimisation iteratively by considering one frequency bin $1 \leq \omega \leq \frac{L}{2}$ at a time and choosing a permutation (in a combinatorial fashion) that minimises the ℓ^p norm locally, while keeping the other bins fixed. To preserve $a_{ij} = \bar{a}_{ij}$, the same permutation is applied on the corresponding mirror frequency $L + 1 - \omega$. This iterative procedure is repeated over all frequency bins till the ℓ^p norms of the filters converges.

We conservatively consider the filters as successfully recovered when the SNR of the permutation corrected time-domain filters exceeds 200dB. Fig. 1 shows the phase transition diagram for filter recovery using the proposed algorithm for the number of sources $N = 4$, number of channels $M = 3$, length of individual filters $L = 1024$ and $p = 1$. White indicates guaranteed success, black is guaranteed failure.

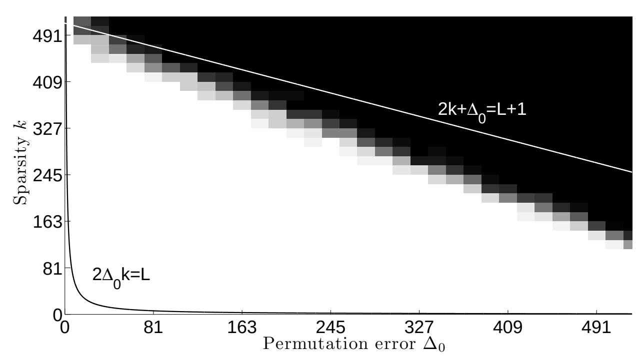

Fig. 1. Phase transition diagram for filter recovery by ℓ^1 minimisation.

The guarantees of Theorem 2 are delimited by the black line in general, and the white line if L is prime. We observe a phase transition close to the prime length case.

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